Solución

Calcule las siguientes integrales indefinidas:

**1.** (7 ptos.) $\displaystyle \int \frac{2e^{2x}-e^{x}}{\sqrt{3e^{2x}-6e^{x}-1}}\,\mathrm{d}x$

$$\int \frac{2e^{2x}-e^{x}}{\sqrt{3e^{2x}-6e^{x}-1}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{3}} \int \frac{2(e^{x}-1)+1}{\sqrt{\left(e^{x}-1\right)^{2}-\left({}^{2}\!/\!\sqrt{3}\right)^{2}}}\; e^{x}\,\mathrm{d}x$$

$$u = e^{x}-1 \quad \Rightarrow \quad \mathrm{d}u = e^{x}\,\mathrm{d}x$$

$$\overset{\downarrow}{=} \frac{1}{\sqrt{3}} \int \frac{2u+1}{\sqrt{u^{2}-\left({}^{2}\!/\!\sqrt{3}\right)^{2}}}\,\mathrm{d}u$$

$$= \frac{1}{\sqrt{3}} \int \frac{2u}{\sqrt{u^{2}-\left({}^{2}\!/\!\sqrt{3}\right)^{2}}}\,\mathrm{d}u + \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{u^{2}-\left({}^{2}\!/\!\sqrt{3}\right)^{2}}}\,\mathrm{d}u$$

$$= \frac{1}{\sqrt{3}} \overbrace{\int \frac{1}{\sqrt{w}}\,\mathrm{d}w}^{\substack{w = u^{2}-\left({}^{2}\!/\!\sqrt{3}\right)^{2} \quad \Rightarrow \quad \mathrm{d}w = 2u\,\mathrm{d}u \\ \downarrow}} + \frac{1}{\sqrt{3}} \underbrace{\int \sec(\theta)\,\mathrm{d}\theta}_{\substack{\uparrow \\ u=\frac{2}{\sqrt{3}}\sec(\theta) \quad \Rightarrow \quad \mathrm{d}u=\frac{2}{\sqrt{3}}\sec(\theta)\tan(\theta)\,\mathrm{d}\theta}}$$

$$= \frac{2}{\sqrt{3}}\sqrt{w} + \frac{1}{\sqrt{3}} \ln\left|\sec(\theta)+\tan(\theta)\right| + C$$

$$= \frac{2}{3}\sqrt{3e^{2x}-6e^{x}-1} + \frac{1}{\sqrt{3}} \ln\left|e^{x}-1+\sqrt{\left(e^{x}-1\right)^{2}-{}^{4}\!/\!{}_{3}}\right| + C$$

**2.** (7 ptos.) $\displaystyle\int e^{ax}\cos(bx)\,\mathrm{d}x$

$$
\begin{aligned}
\int e^{ax}\cos(bx)\,\mathrm{d}x \\
&= \frac{1}{a}\,e^{ax}\cos(bx) - \int \frac{1}{a}\,e^{ax}\Big(-b\,\mathrm{sen}(bx)\Big)\,\mathrm{d}x \\
&= \frac{1}{a}\,e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\,\mathrm{sen}(bx)\,\mathrm{d}x \\
&= \frac{1}{a}\,e^{ax}\cos(bx) + \frac{b}{a}\left(\frac{1}{a}\,e^{ax}\,\mathrm{sen}(bx) - \int \frac{1}{a}\,e^{ax}\Big(b\cos(bx)\Big)\,\mathrm{d}x\right) \\
&= \frac{1}{a}\,e^{ax}\cos(bx) + \frac{b}{a^2}\,e^{ax}\,\mathrm{sen}(bx) - \frac{b^2}{a^2}\int e^{ax}\cos(bx)\,\mathrm{d}x
\end{aligned}
$$

de donde,

$$\int e^{ax}\cos(bx)\,\mathrm{d}x = \frac{1}{a^2+b^2}\Big(a\,e^{ax}\cos(bx) + b\,e^{ax}\,\mathrm{sen}(bx)\Big) + C$$

Equivalentemente,

$$
\begin{aligned}
\int e^{ax}\cos(bx)\,\mathrm{d}x \\
&= \frac{1}{b}\,e^{ax}\,\mathrm{sen}(bx) - \frac{a}{b}\int e^{ax}\,\mathrm{sen}(bx)\,\mathrm{d}x \\
&= \frac{1}{b}\,e^{ax}\,\mathrm{sen}(bx) - \frac{a}{b}\left(\frac{1}{b}\,e^{ax}\Big(-\cos(bx)\Big) - \frac{a}{b}\int e^{ax}\Big(-\cos(bx)\Big)\,\mathrm{d}x\right) \\
&= \frac{1}{b}\,e^{ax}\,\mathrm{sen}(bx) + \frac{a}{b^2}\,e^{ax}\cos(bx) - \frac{a^2}{b^2}\int e^{ax}\cos(bx)\,\mathrm{d}x
\end{aligned}
$$

de donde,

$$\int e^{ax}\cos(bx)\,\mathrm{d}x = \frac{1}{a^2+b^2}\Big(a\,e^{ax}\cos(bx) + b\,e^{ax}\,\mathrm{sen}(bx)\Big) + C$$

**3.** (7 ptos.) $\displaystyle\int \frac{1}{x+\sqrt{x^2+1}}\,\mathrm{d}x$

$$
\begin{aligned}
&\int \frac{1}{x+\sqrt{x^2+1}}\,\mathrm{d}x \\
&\qquad = \int \frac{1}{x+\sqrt{x^2+1}}\left(\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right)\mathrm{d}x \\
&\qquad = \int \left(\sqrt{x^2+1}-x\right)\mathrm{d}x \\
&\qquad = \int \sqrt{x^2+1}\,\mathrm{d}x - \int x\,\mathrm{d}x
\end{aligned}
$$

$$x=\tan(\theta) \quad \Rightarrow \quad \sqrt{x^2+1}=\sec(\theta) \ \text{ y } \ \mathrm{d}x=\sec^2(\theta)$$

$$
\begin{aligned}
&\qquad = \overbrace{\int \sec^3(\theta)\,\mathrm{d}x}^{\downarrow} - \frac{1}{2}\,x^2 \\
&\qquad = \int \sec(\theta)\sec^2(\theta)\,\mathrm{d}x - \frac{1}{2}\,x^2 \\
&\qquad = \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)\,\mathrm{d}x - \frac{1}{2}\,x^2 \\
&\qquad = \sec(\theta)\tan(\theta) - \int \sec(\theta)\left(\sec^2(\theta)-1\right)\mathrm{d}x - \frac{1}{2}\,x^2 \\
&\qquad = \sec(\theta)\tan(\theta) - \int \sec^3(\theta)\,\mathrm{d}x + \int \sec(\theta)\,\mathrm{d}x - \frac{1}{2}\,x^2 \\
&\qquad = \sec(\theta)\tan(\theta) - \int \sec^3(\theta)\,\mathrm{d}x + \ln\left|\sec(\theta)+\tan(\theta)\right| - \frac{1}{2}\,x^2 \\
&\qquad = \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln\left|\sec(\theta)+\tan(\theta)\right| - \frac{1}{2}\,x^2 + C \\
&\qquad = \frac{1}{2}\,x\sqrt{x^2+1} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| - \frac{1}{2}\,x^2 + C
\end{aligned}
$$

**4.** (7 ptos.) $\displaystyle\int \frac{-x^4+x^2+5x+1}{x^5-x^3-x^2+1}\,\mathrm{d}x$

$$
\begin{aligned}
\frac{-x^4+x^2+5x+1}{x^5-x^3-x^2+1} &= \frac{-x^4+x^2+5x+1}{(x+1)(x-1)^2(x^2+x+1)} \\
&= \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{Dx+E}{x^2+x+1}
\end{aligned}
$$

$$
\begin{aligned}
-x^4+x^2+5x+1 \quad &= \quad A(x-1)^2(x^2+x+1) \\
&\quad +B(x+1)(x-1)(x^2+x+1) \\
&\quad +C(x+1)(x^2+x+1) \\
&\quad +(Dx+E)(x+1)(x-1)^2
\end{aligned}
$$

$$
\begin{aligned}
x=-1 \quad &\Rightarrow \quad A=-1 \\
x=1 \quad &\Rightarrow \quad C=1
\end{aligned}
$$

$$
\begin{aligned}
\overset{\downarrow}{=} \quad & (-1+B+D)x^4 \\
& +(2+B-D+E)x^3 \\
& +(2-D-E)x^2 \\
& +(3-B+D-E)x \\
& +(-B+E)
\end{aligned}
$$

$$
\left.
\begin{aligned}
-1+B+D&=-1 \\
2+B-D+E&=0 \\
2-D-E&=1 \\
3-B+D-E&=5 \\
-B+E&=1
\end{aligned}
\right\}
\quad \Longrightarrow \quad B=-1, \quad D=1, \quad E=0
$$

$$\int \frac{-x^4+x^2+5x+1}{x^5-x^3-x^2+1}\,\mathrm{d}x$$

$$= \int \left( \frac{-1}{x+1} + \frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{x}{x^2+x+1} \right) \mathrm{d}x$$

$$= -\ln|x+1| - \ln|x-1| - \frac{1}{x-1} + \frac{1}{2} \int \frac{2x+1-1}{x^2+x+1}\,\mathrm{d}x$$

$$= -\ln|x+1| - \ln|x-1| - \frac{1}{x-1} + \frac{1}{2} \int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x$$
$$-\frac{1}{2}\int \frac{1}{x^2+x+1}\,\mathrm{d}x$$

$$= -\ln|x+1| - \ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln\left|x^2+x+1\right|$$
$$-\frac{1}{2}\int \frac{1}{x^2+x+1}\,\mathrm{d}x$$

$$= \ln\left|\frac{\sqrt{x^2+x+1}}{(x+1)(x-1)}\right| - \frac{1}{x-1} - \frac{1}{2}\int \frac{1}{\left(x+{}^1\!/_2\right)^2 + \left(\sqrt{3}/2\right)^2}\,\mathrm{d}x$$

$$= \ln\left|\frac{\sqrt{x^2+x+1}}{(x+1)(x-1)}\right| - \frac{1}{x-1}$$
$$-\frac{1}{\sqrt{3}} \int \frac{1}{\left(\frac{2}{\sqrt{3}}\left(x+{}^1\!/_2\right)\right)^2+1} \left(\frac{2}{\sqrt{3}}\right) \mathrm{d}x$$

$$u = \frac{2}{\sqrt{3}}\left(x+{}^1\!/_2\right) \quad \Rightarrow \quad \mathrm{d}u = \frac{2}{\sqrt{3}}\,\mathrm{d}x$$
$$\downarrow$$

$$= \ln\left|\frac{\sqrt{x^2+x+1}}{(x+1)(x-1)}\right| - \frac{1}{x-1} - \frac{1}{\sqrt{3}} \overbrace{\int \frac{1}{u^2+1}\,\mathrm{d}u}$$

$$= \ln\left|\frac{\sqrt{x^2+x+1}}{(x+1)(x-1)}\right| - \frac{1}{x-1} - \frac{1}{\sqrt{3}} \arctan(u) + C$$

$$= \ln\left|\frac{\sqrt{x^2+x+1}}{(x+1)(x-1)}\right| - \frac{1}{x-1} - \frac{1}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(x+{}^1\!/_2\right)\right) + C$$

**5.** (3 ptos.) $\displaystyle\int \frac{1}{1+e^x}\,\mathrm{d}x$

$$\int \frac{1}{1+e^x}\,\mathrm{d}x \;=\; \int \frac{1+e^x-e^x}{1+e^x}\,\mathrm{d}x \;=\; \int \left(1-\frac{e^x}{1+e^x}\right)\mathrm{d}x$$

$$= \int \mathrm{d}x - \int \frac{e^x}{1+e^x}\,\mathrm{d}x \;=\; x-\ln(1+e^x)+C$$

Equivalentemente,

$$\int \frac{1}{1+e^x}\,\mathrm{d}x \;=\; -\int \frac{-e^{-x}}{e^{-x}+1}\,\mathrm{d}x \;=\; -\ln\big(e^{-x}+1\big)+C$$

$$= \ln\left(\frac{1}{e^{-x}+1}\right)+C \;=\; \ln\left(\frac{e^x}{1+e^x}\right)+C$$

$$= \ln(e^x)-\ln(1+e^x)+C \;=\; x-\ln(1+e^x)+C$$

**6.** (3 ptos.) $\displaystyle\int \sqrt{e^x+1}\,\mathrm{d}x$

$$u^2=e^x+1 \quad\Rightarrow\quad 2\,\frac{u}{u^2-1}\,\mathrm{d}u=\mathrm{d}x$$

$$\int \sqrt{e^x+1}\,\mathrm{d}x \;\overset{\downarrow}{=}\; 2\int \frac{u^2}{u^2-1}\,\mathrm{d}u = 2\int \frac{u^2-1+1}{u^2-1}\,\mathrm{d}u$$

$$= 2\int \left(1+\frac{1}{u^2-1}\right)\mathrm{d}u$$

$$= 2\int \left(1+\frac{1}{2}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)\right)\mathrm{d}u$$

$$= 2\int \mathrm{d}u+\int \frac{1}{u-1}\,\mathrm{d}u-\int \frac{1}{u+1}\,\mathrm{d}u$$

$$= 2u+\ln|u-1|-\ln|u+1|+C$$

$$= 2u+\ln\left|\frac{u-1}{u+1}\right|+C$$

$$= 2\sqrt{e^x+1}+\ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right|+C$$

**7.** (3 ptos.) $\displaystyle\int \frac{1}{x^2\sqrt{a^2-x^2}}\,\mathrm{d}x$

$$\int \frac{1}{x^2\sqrt{a^2-x^2}}\,\mathrm{d}x \overset{\substack{x = a\operatorname{sen}(\theta) \;\Rightarrow\; \mathrm{d}x = a\cos(\theta)\,\mathrm{d}\theta \\ \downarrow}}{=} \frac{1}{a^2}\int \csc^2(\theta)\,\mathrm{d}\theta = -\frac{1}{a^2}\operatorname{cotan}(\theta) + C$$

$$= -\left(\frac{1}{a^2}\right)\frac{\sqrt{a^2-x^2}}{x} + C$$

Equivalentemente,

$$\int \frac{1}{x^2\sqrt{a^2-x^2}}\,\mathrm{d}x \overset{\substack{x = a\cos(\theta) \;\Rightarrow\; \mathrm{d}x = -a\operatorname{sen}(\theta)\,\mathrm{d}\theta \\ \downarrow}}{=} -\frac{1}{a^2}\int \sec^2(\theta)\,\mathrm{d}\theta = -\frac{1}{a^2}\tan(\theta) + C$$

$$= -\left(\frac{1}{a^2}\right)\frac{\sqrt{a^2-x^2}}{x} + C$$

**8.** (3 ptos.) $\displaystyle\int \frac{1}{x\ln(x^n)}\,\mathrm{d}x$

$$\int \frac{1}{x\ln(x^n)}\,\mathrm{d}x = \int \frac{1}{x\Big(n\ln(x)\Big)}\,\mathrm{d}x = \frac{1}{n}\int \frac{1/x}{\ln(x)}\,\mathrm{d}x = \frac{1}{n}\ln\Big|\ln(x)\Big| + C$$